Random moments for the new eigenfunctions of point scatterers on rectangular flat tori

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Point scatterers and their eigenfunctions

Spectrum of rectangular flat tori

$\mathbb{T}_\alpha = \mathbb{R}^2/\left(\alpha\mathbb{Z} \oplus \frac{1}{\alpha}\mathbb{Z}\right)$, with $\alpha > 0$.

$$\mathrm{Sp}(\Delta) = \left\{ \frac{4\pi^2}{\alpha^2}(a^2 + \alpha^4 b^2) \;\middle|\; a, b \in \mathbb{Z} \right\} = \{\lambda_k \mid k \geqslant 0\},$$

where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \xrightarrow[k \to +\infty]{} +\infty$.

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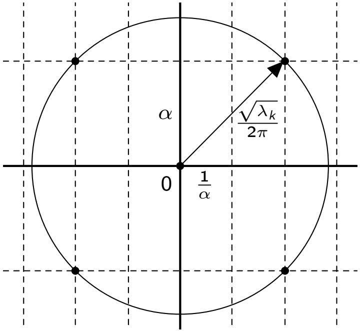
where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k < \cdots \xrightarrow[k \to +\infty]{} +\infty$.

$\Lambda_k = \left\{ \xi \in \frac{1}{\alpha}\mathbb{Z} \oplus \alpha\mathbb{Z} \;\middle|\; \|\xi\| = \frac{\sqrt{\lambda_k}}{2\pi} \right\}$
wave vectors associated with λ_k.

$\ker(\Delta - \lambda_k) = \mathrm{Span}\left\{ e^{2i\pi\langle \xi\, ,\cdot \rangle} \;\middle|\; \xi \in \Lambda_k \right\}$.

If $k \geqslant 1$, then $|\Lambda_k| \geqslant 2$;
and generically $|\Lambda_k| \geqslant 4$.



Point scatterers

Informally a point scatterer is "$\Delta + \delta$", where $(\Delta + \delta)f = \Delta f + f(0)\delta$.

Theorem (von Neumann)

Let $D_0 = \{f \in \mathcal{C}^\infty(\mathbb{T}_\alpha) \mid f \text{ vanishes in a neighborhood of } 0\}$.
There is a family $(\Delta_\varphi)_{\varphi \in (-\pi,\pi]}$ of self-adjoint extensions of $\Delta_{|D_0}$ to $L^2(\mathbb{T}_\alpha)$.

When $\varphi = \pi$, we recover $\Delta_\varphi = \Delta$.

A point scatterer is Δ_φ with $\varphi \in (-\pi, \pi)$.

Spectrum of a point scatterer

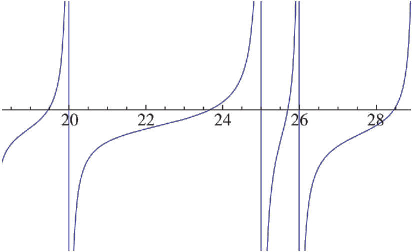
$$\mathrm{Sp}(\Delta_\varphi) = \{\lambda_k \mid k \geqslant 1\} \sqcup \left\{\tau_k^\varphi \mid k \geqslant 0\right\}.$$

For $k \geqslant 1$, $\ker(\Delta_\varphi - \lambda_k) = \{\phi \in \ker(\Delta - \lambda_k) \mid \phi(0) = 0\}$.

$\left(\tau_k^\varphi\right)_{k\geqslant 0}$ are new simple eigenvalues, solutions of

$$\Psi(\tau) = \tan\left(\frac{\varphi}{2}\right),$$

where Ψ rational with poles $(\lambda_k)_{k\geqslant 0}$.



New eigenfunctions of Δ_φ on $\mathbb{T}_\alpha$

For $\tau \in \mathbb{R} \setminus \{\lambda_k \mid k \geqslant 0\}$, we define

$$f_\tau(x) = \sum_{k \geqslant 1} \frac{1}{\lambda_k - \tau} \sum_{\xi \in \Lambda_k} e^{2i\pi\langle \xi , \cdot \rangle}.$$

If τ is a new eigenvalue of Δ_φ, then $f_\tau - \frac{1}{\tau}$ is an associated eigenfunction.

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If τ is a new eigenvalue of Δ_φ, then $f_\tau - \frac{1}{\tau}$ is an associated eigenfunction.

- The new eigenfunctions are reasonnably explicit.
- For well-chosen values of α, they exhibit quantum chaotic features.

Theorem (Kurlberg–Ueberschär, 2014)

If $\alpha = 1$, there is a full-density subsequence of new eigenfunctions which is quantum ergodic. In particular, their L^2-mass equidistributes as $\tau_k^\varphi \to +\infty$.

The Seba Conjecture

Let X uniform random point in $\mathbb{T}_\alpha$, we consider the random variable $f_\tau(X)$.

$$\mathbb{E}[f_\tau(X)] = \int_{\mathbb{T}_\alpha} f_\tau(x)\,\mathrm{d}x = 0$$

$$\mathrm{Var}(f_\tau(X)) = \int_{\mathbb{T}_\alpha} f_\tau(x)^2\,\mathrm{d}x = \sum_{k\geqslant 1} \frac{|\Lambda_k|}{(\lambda_k - \tau)^2}.$$

More generally, for $p \in \mathbb{N}^*$, $M_p(\tau) = \int_{\mathbb{T}_\alpha} f_\tau(x)^p\,\mathrm{d}x$.

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Conjecture (Seba, 1990)

For all $p \geqslant 3$, $\quad \dfrac{M_p(\tau_k^\varphi)}{M_2(\tau_k^\varphi)^{\frac{p}{2}}} \xrightarrow[k\to+\infty]{} \mu_p = \begin{cases} (p-1)(p-3)\cdots 1 & \textit{if } p \textit{ even}, \\ 0 & \textit{if } p \textit{ odd}. \end{cases}$

Implies a Central Limit Theorem for the value distribution $f_{\tau_k^\varphi}(X)$.

Seba's Conjecture is false (sometimes)

- Keating–Marklov–Winn (2003): non-Gaussian limit for some α.
- Kurlberg–Ueberschär (2019): if α^4 is diophantine then

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From now on:

- forget about point scatterers and φ,
- we study the moments of f_τ as $\tau \to +\infty$.

Poisson processes and the Berry–Tabor Conjecture

Poisson point process

Let ν be a Borel measure on $[0, +\infty)$ such that:

- ν admits a bounded positive density,
- $\nu([0, +\infty)) = +\infty$.

Poisson point process with intensity ν

$P \subset [0, +\infty)$ a random subset such that:

- for any interval $I \subset [0, +\infty)$, we have $|P \cap I| \sim \propto(\nu(I))$
- if $I_1, \ldots, I_m$ are disjoint intervals then $(|P \cap I_i|)_{1 \leqslant i \leqslant m}$ are independent.

Almost surely $P = \{\lambda_k \mid k \geqslant 1\}$, where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \xrightarrow[k \to +\infty]{} +\infty.$$

A useful tool: Campbell's Theorem

Let g be a non-negative Borel function and $S = \sum_{k \geqslant 1} g(\lambda_k)$.

1. We have $\mathbb{E}[S] = \int_0^{+\infty} g \, \mathrm{d}\nu$.

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2. If $\mathbb{E}[S] < +\infty$, then $\mathrm{Var}(S) = \int_0^{+\infty} g^2 \, \mathrm{d}\nu$.

3. If $\int_0^{+\infty} \min(g, 1) \, \mathrm{d}\nu < +\infty$, then

$$\forall x \in \mathbb{R}, \qquad \ln\left(\mathbb{E}\left[e^{ixS}\right]\right) = \int_0^{+\infty} \left(e^{ixg(t)} - 1\right) \mathrm{d}\nu(t).$$

The Berry–Tabor Conjecture

Conjecture (Berry–Tabor, 1977)

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Poisson point processes with constant intensity $\beta > 0$

$(\lambda_k)_{k\geqslant 1}$ is a Poisson point process with intensity $\nu = \beta \, dx$. $\iff$ $(\lambda_k - \lambda_{k-1})_{k\geqslant 1}$ are independent $\mathcal{E}xp(\beta)$ variables, with $\lambda_0 = 0$.

In this case, almost surely, $\frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k - \lambda_{k-1}} \xrightarrow[n\to+\infty]{\text{law}} \mathcal{E}xp(\beta)$

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- If $\alpha^4 \notin \mathbb{Q}$, numerically $\frac{1}{n}\sum_{k=1}^{n} \delta_{\lambda_k - \lambda_{k-1}} \xrightarrow[n\to+\infty]{\text{law}} \mathcal{E}\mathrm{xp}\left(\frac{1}{16\pi}\right)$.
- If α^4 is diophantine, $\frac{1}{n}\sum_{1\leqslant k,l\leqslant n} \delta_{\lambda_k - \lambda_l}$ has a Poissonian limit (Sarnak; Eskin–Margulis–Moses).

Why $\frac{1}{16\pi}$? The Weyl Law!

Laplace spectrum on $\mathbb{T}_\alpha$ with $\alpha^4 \notin \mathbb{Q}$

For all $k \geqslant 1$ we have $|\Lambda_k| \in \{2, 4\}$, and generically $|\Lambda_k| = 4$.

$$\big|\operatorname{Sp}(\Delta) \cap (0, \lambda]\big| \simeq \frac{1}{4} \sum_{\lambda_k \leqslant \lambda} |\Lambda_k| =: \frac{1}{4} \mathcal{N}(\lambda) \sim \frac{\lambda}{16\pi}.$$

Poisson point process with intensity $\nu = \beta \, \mathrm{d}x$

We have $\nu([0, \lambda]) = \beta\lambda$, hence $\big|\{k \geqslant 1 \mid \lambda_k \leqslant \lambda\}\big| \sim \propto(\beta\lambda)$ and

$$\mathbb{E}\Big[\big|\{k \geqslant 1 \mid \lambda_k \leqslant \lambda\}\big|\Big] = \beta\lambda.$$

With $\beta = \frac{1}{16\pi}$, the Poisson process satisfies the Weyl Law on average.

A simple plan

f_τ only depends on τ and the sequences $\underline{\lambda} = (\lambda_k)_{k \geqslant 1}$ and $\underline{\Lambda} = (\Lambda_k)_{k \geqslant 1}$.

- Replace $\underline{\lambda}$ with a Poisson point process.
- Tune its intensity ν in order to agree with the Weyl Law.
- Choose random directions for the wave vectors in $\Lambda_k \cap [0, +\infty)^2$ and add symmetric vectors (for example: independent uniform directions).

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Problems

- f_τ no longer defines a function on $\mathbb{T}_\alpha$.
- Subtle interactions between ν and the multiplicities $(|\Lambda_k|)_{k\geqslant 1}$.

A random model for the moments of the new eigenfunctions

Step 1: deterministic expression of the moments

ℓ_0 space of sequences $a = (a_k)_{k \geqslant 1}$ with values in $\mathbb{N}$ and finite support.

$$|a| = \sum_{k \geqslant 1} a_k \qquad \text{and} \qquad a! = \prod_{k \geqslant 1} a_k!.$$

Definition

Let $a \in \ell_0$, $N_a(\underline{\Lambda}) = \left| \left\{ (\xi_{k,l})_{k \geqslant 1; 1 \leqslant l \leqslant a_k} \;\middle|\; \xi_{k,l} \in \Lambda_k \text{ and } \sum_{k \geqslant 1} \sum_{l=1}^{a_k} \xi_{k,l} = 0 \right\} \right|$.

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Lemma

$$M_p(\tau) = p! \sum_{\{a \in \ell_0 \,|\, |a| = p\}} \frac{N_a(\underline{\Lambda})}{a!} \prod_{k \geqslant 1} \left(\frac{1}{\lambda_k - \tau} \right)^{a_k}$$

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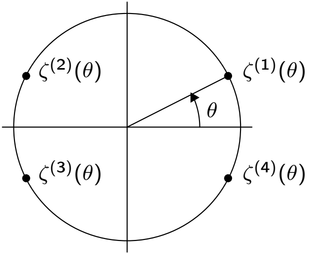
Step 2: randomization of the wave vector sets

$\underline{\lambda} = (\lambda_k)_{k \geqslant 1}$ positive increasing sequence, $\underline{m} = (m_k)_{k \geqslant 1}$ positive integers.

$\underline{\theta} = (\theta_{k,l})_{k,l \geqslant 1}$ random sequence in $[0, \frac{\pi}{2}]$, admits a density with respect to Lebesgue.

Randomized wave vector sets

$$\Lambda_k = \frac{\sqrt{\lambda_k}}{2\pi} \left\{ \zeta^{(i)}(\theta_{k,l}) \;\middle|\; \begin{array}{l} 1 \leqslant i \leqslant 4 \\ 1 \leqslant l \leqslant m_k \end{array} \right\}.$$



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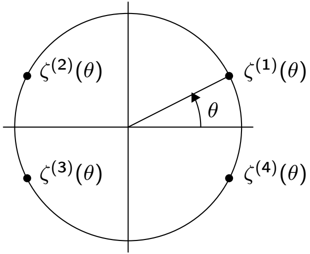

Examples of distribution of $(\theta_{k,l})_{k,l \geqslant 1}$

- Uniform and independent.
- Localized in small intervals.
- Repulsion between the directions associated with λ_k.

Almost surely, for all $k \geqslant 1$, $|\Lambda_k| = 4m_k$.

Almost sure expression of the randomized moments

Lemma

Almost surely, $\forall a \in \ell_0$, $N_a(\underline{\Lambda}) = N_a^0(\underline{m})$ *(depends only on a and* $\underline{m}$*).*

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Lemma

Almost surely, $\forall a \in \ell_0$, $N_a(\underline{\Lambda}) = N_a^0(\underline{m})$ (depends only on a and $\underline{m}$).

Almost surely,

$$F_p\Big(\tau, \underline{\lambda}, (N_a(\underline{\Lambda}))_{|a|=p}\Big) = F_p\Big(\tau, \underline{\lambda}, \big(N_a^0(\underline{m})\big)_{|a|=p}\Big)$$
$$= \begin{cases} 0 \ = \mu_p & \text{if } p \text{ odd,} \\ P_p\big(S_1(\tau, \underline{\lambda}, \underline{m}), \ldots, S_p(\tau, \underline{\lambda}, \underline{m})\big) & \text{if } p \text{ even,} \end{cases}$$

where $P_p(X_1, \ldots, X_p)$ explicit polynomial of degree p and

$$S_j(\tau, \underline{\lambda}, \underline{m}) = \sum_{k \geqslant 1} \frac{m_k}{(\lambda_k - \tau)^{2j}}.$$

Step 3: randomization of the Laplace spectrum

Randomized multiplicities and spectrum

Let $m : [0, +\infty) \to [1, +\infty)$ such that $m(t) = o(t)$,

- $\underline{\lambda} = (\lambda_k)_{k \geqslant 1}$ Poisson point process with intensity $\nu_m = \frac{1}{16\pi m} \mathrm{d}x$,
- for all $k \geqslant 1$, $m_k = m(\lambda_k)$.

We set $\mathcal{N}_m(\lambda) = \sum_{\lambda_k \leqslant \lambda} 4m_k$.

$$\mathbb{E}[\mathcal{N}_m(\lambda)] = \mathbb{E}\left[\sum_{k \geqslant 1} 4m(\lambda_k)\mathbf{1}_{[0,\lambda]}(\lambda_k)\right] = \int_0^\lambda 4m(t)\,\mathrm{d}\nu_m(t) = \frac{\lambda}{4\pi}.$$

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Let $\tau \in \mathbb{R}^*$, almost surely $\tau \notin \{\lambda_k \mid k \geqslant 1\}$ and $S_j(\tau, \underline{\lambda}, \underline{m}) < +\infty$.

Campbell's Theorem yields the characteristic function of $S_j(\tau, \underline{\lambda}, \underline{m})$.

Choice of the multiplicity function

Multiplicity function

m of class $\mathcal{C}^1$ and slowly varying: $m'(t) = O(t^{-\beta})$ for some $\beta > 0$.

Relevant examples

- $m : t \mapsto 1$ (generic multiplicities on $\mathbb{T}_\alpha$ with $\alpha^4 \notin \mathbb{Q}$).
- $m(t) \sim C\sqrt{\ln(t)}$ (average growth of multiplicities on $\mathbb{T}_1$).

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Lemma

$$m(\tau)^{2j-1} S_j(\tau, \underline{\lambda}, \underline{m}) \xrightarrow[\tau \to +\infty]{law} S_j,$$

S_j random variable with explicit characterictic function, depends only on j.

Limit distribution of the randomized moments

m multiplicity function, $\left(M_{2p}(\tau)\right)_{p\geqslant 1}$ associated even randomized moments.

Theorem (L.–Ueberschär, 2021)

There exists a family $\left\{R_p(\ell) \;\middle|\; p \geqslant 2 \text{ and } \ell \in [1,+\infty]\right\}$ *of random variables such that:*

$$\text{if} \qquad m(\tau) \xrightarrow[\tau\to+\infty]{} \ell \qquad \text{then} \qquad \frac{M_{2p}(\tau)}{M_2(\tau)^p} \xrightarrow[\tau\to+\infty]{\text{law}} \mu_{2p} R_p(\ell).$$

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Same along sequences $\tau_k \xrightarrow[k\to+\infty]{} +\infty$ *such that* $m(\tau_k) \xrightarrow[k\to+\infty]{} \ell$.

Moreover,

- $R_p(\ell) = R_p(\ell')$ *in distribution if and only if* $\ell = \ell'$.
- $R_p(+\infty) = 1$ *deterministically.*
- *If* $\ell < +\infty$, *then* $R_p(\ell)$ *admits a smooth density.*

Bonus

The Weyl Law in the random model

$(\lambda_k)_{k\geqslant 1}$ Poisson point process with intensity $\nu_m = \frac{1}{16\pi m}\,\mathrm{d}x$ and

$$\mathcal{N}_m(\lambda) = \sum_{\lambda_k \leqslant \lambda} 4m_k = \sum_{k\geqslant 1} 4m_k(\lambda_k)\mathbf{1}_{[0,\lambda]}(\lambda_k).$$

Recall that $\mathbb{E}[\mathcal{N}_m(\lambda)] = \frac{\lambda}{4\pi}$.

Theorem (L.–Ueberschär, 2021)

- $\dfrac{\lambda}{\pi} \leqslant \mathrm{Var}(\mathcal{N}_m(\lambda)) = \dfrac{1}{\pi}\displaystyle\int_0^\lambda m(t)\,\mathrm{d}t = O\left(\lambda^{1+\alpha}\right)$, *for some* $\alpha \in [0,1)$.

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- *Law of Large Numbers:* $\frac{1}{\lambda}\mathcal{N}_m(\lambda) \xrightarrow[\lambda\to+\infty]{\text{almost sure}} \frac{1}{4\pi}$.

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- *Law of Large Numbers:* $\frac{1}{\lambda}\mathcal{N}_m(\lambda) \xrightarrow[\lambda\to+\infty]{\text{almost sure}} \frac{1}{4\pi}$.
- *Central Limit Theorem:* $\dfrac{\mathcal{N}_m(\lambda) - \frac{\lambda}{4\pi}}{\sqrt{\frac{1}{\pi}\int_0^\lambda m(t)\,\mathrm{d}t}} \xrightarrow[\lambda\to+\infty]{\text{law}} \mathcal{N}(0,1)$.

The end

Thank you for your attention.